
PYHS-201(e)
General Physics: Electromagnetism
Prof. P. Scarlino
Mock Exam
December 23, 2024

Name: _____

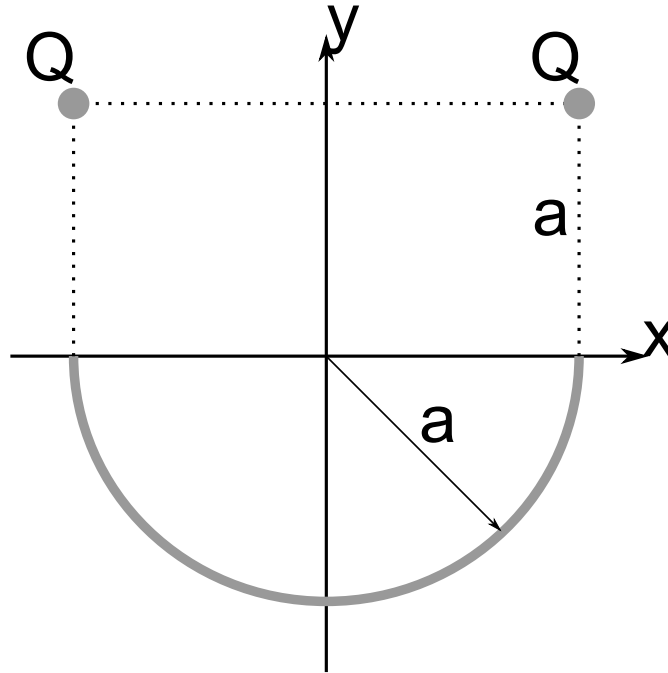
Sciper: _____

- The exam takes place from 9:15 to 12:15 a.m.
- Please put your valid ID on the table.
- There are five problems. Each problem provides different points as indicated. The total number of points is 45. Check that you have received all problems.
- One page of handwritten notes is allowed as well as a calculator without equation solver. The items will be checked in the course of the written exam.
- If you want to leave before the end of the written exam, please raise your hand and wait until an assistant arrives at your place.
- Please respect your colleagues and do not leave between 11:45 and 12:15 am. Use remaining time to check your solutions.
- Please remain seated while the papers are collected at the end of the exam.
- The copy must be written in blue or black pen or fountain pen, using indelible ink; pencil, in particular, is not allowed, except for graphs.
- **Put your name and Sciper number on every paper sheet (like the sticker in the first page).**
- Number each page and indicate the total number of double sheets on the first page at the end of the exam.
- For a graph to be considered correct, it must have a legend, the axes must be named and, if it makes sense, scaled.
- **Please give at least a brief explanation of the calculations you are undertaking. Any answer requires a justification, even a brief one.**
- Useful constants: $\varepsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1}$, $\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$, $c = 3 \times 10^8 \text{ m s}^{-1}$.

Problem 1:

(6 pts)

Consider the charge distribution shown in the figure below. A semicircle of radius a centered at the origin (indicated by the gray curved line) carries a uniform positive charge per unit length λ . Two positive point charges Q are located at points $(-a, a)$ and (a, a) (indicated by the two gray dots).



1. Find the expression of the electric field at the origin due to the charged semicircle only. Express your answer in unit vector notation.

We can split the arc into individual point charges and sum up the electric field they generate by integrating along the arc. The magnitude of each field element is given by

$$dE = k \frac{dq}{r^2} = k \frac{\lambda ds}{a^2} = k \frac{\lambda a d\theta}{a^2} = k \frac{\lambda d\theta}{a}. \quad (1)$$

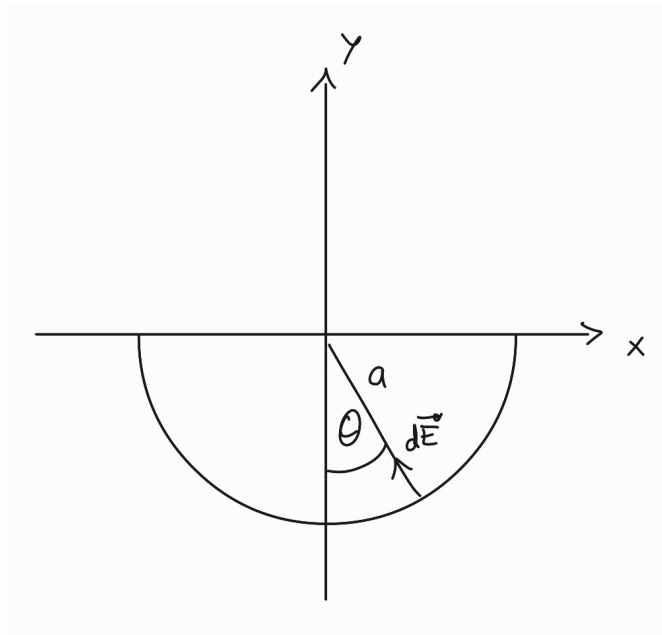
Here, $ds = a \cdot d\theta$ is the length of the charge element dq and θ is the angle between the charge element and the y axis (see figure below). Again, due to symmetry, the electric field is given purely by its y component.

$$dE_y = dE \cos(\theta) = k \frac{\lambda}{a} \cos(\theta) d\theta. \quad (2)$$

The total electric field can now be obtained by integrating θ from $-\pi/2$ to $\pi/2$

$$E_y = \frac{k\lambda}{a} \int_{-\pi/2}^{\pi/2} \cos(\theta) d\theta = \frac{k\lambda}{a} \sin \theta \Big|_{-\pi/2}^{\pi/2} = \frac{2k\lambda}{a}. \quad (3)$$

Adding the direction, we have



$$\vec{E} = \frac{2k\lambda}{a} \hat{y} \quad (4)$$

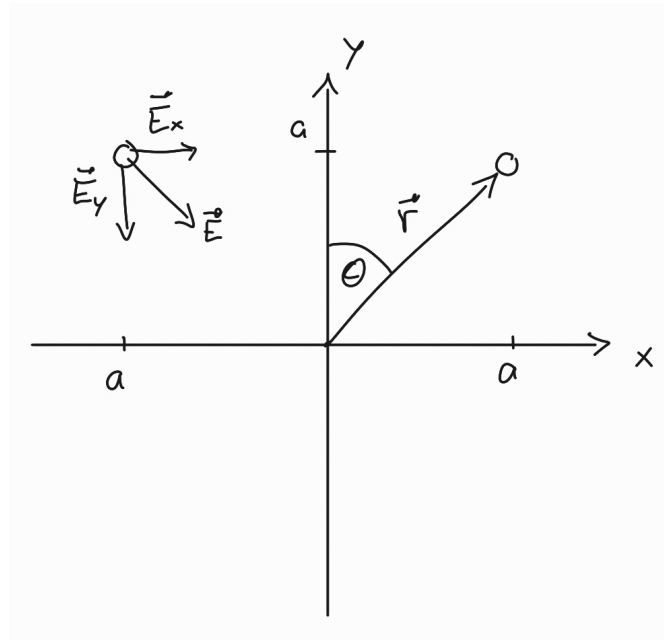
2. Find the expression of the electric field at the origin due to the point charges only. Express your answer in unit vector notation.

Due to the given symmetry, the x components of the electric field of the two point charges compensate each other, so the total electric field is given only by sum of the y components. Considering that the angle θ (see figure below) between the position vectors of the charges and the y axis is $\pm\pi/4$, the total electric field is given by

$$E_y = -\frac{2kQ}{(a\sqrt{2})^2} \cos(\pi/4) = -\frac{kQ}{\sqrt{2}a^2}. \quad (5)$$

Adding the direction, we have

$$\vec{E} = -\frac{kQ}{\sqrt{2}a^2} \hat{y} \quad (6)$$



3. Find the value of Q for which the total electric field at the origin is zero.

The electric field at the origin is zero if the vector sum of the two contributions is zero, which imposes

$$\vec{E} = \vec{E}_{\text{arc}} + \vec{E}_{\text{charge}} = \frac{2k\lambda}{a}\hat{y} - \frac{kQ}{\sqrt{2}a^2}\hat{y} = 0 \quad (7)$$

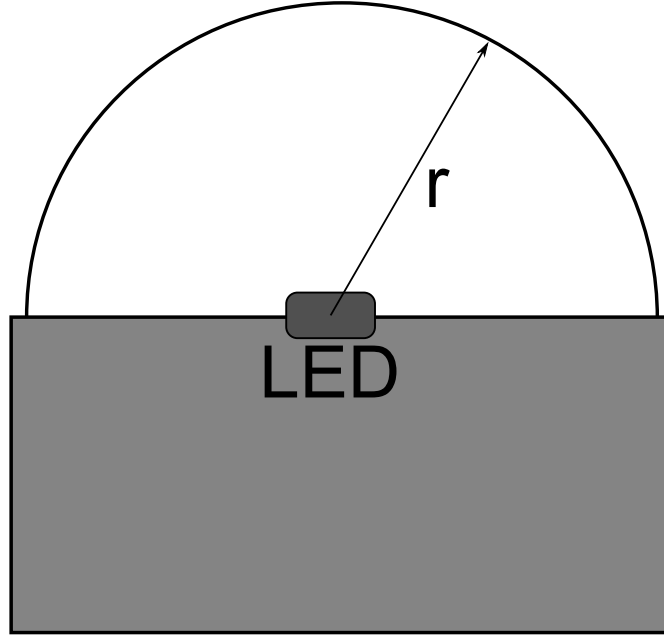
This allows us to obtain the charge Q as

$$\frac{2k\lambda}{a} = \frac{kQ}{\sqrt{2}a^2} \rightarrow Q = 2\sqrt{2}\lambda a. \quad (8)$$

Problem 2:

(5 pts)

A string of Christmas lights consists of LEDs covered by transparent hemispherical domes, as shown in the figure below. The dome has a radius r of 7.50 mm. The LEDs convert electrical energy into light radiation energy at a rate of 0.096 W. The LEDs radiate uniformly into the hemispherical dome.



1. Find the wavenumber k and angular frequency ω of the light knowing that the LEDs emit red light with a wavelength of 650 nm. Give the analytical expressions and numerical answers.

The wave number is defined by $k = 2\pi/\lambda = \frac{2\pi}{650 \times 10^{-9} \text{ m}} = 9.67 \times 10^6 \text{ m}^{-1}$ and the angular frequency is defined by $\omega = 2\pi f = 2\pi \frac{c}{\lambda} = 2.9 \times 10^{15} \text{ rad/s}$.

2. Find the intensity of the radiation at the surface of the hemispherical dome. Give the mathematical expression and numerical answer.

The intensity is given by definition $\mathcal{I} = P/A = P/\frac{1}{2}(4\pi r^2) = 272 \text{ W/m}^2$, where A is the surface of the hemisphere.

3. Find the amplitudes of the electric and magnetic fields at the surface of the hemispherical dome. Give the mathematical expressions and numerical answers.

Alternatively, the intensity can also be given in terms of the maximal amplitude E_{max} of the electric field $\mathcal{I} = \frac{1}{2}c\epsilon_0 E_{\text{max}}^2$. From this we find

$$E_{\text{max}} = \sqrt{\frac{2\mathcal{I}}{c\epsilon_0}} = 453 \text{ V/m.} \quad (9)$$

and the magnetic field

$$B_{\max} = \frac{E_{\max}}{c} = 151 \times 10^{-6} \text{ T}. \quad (10)$$

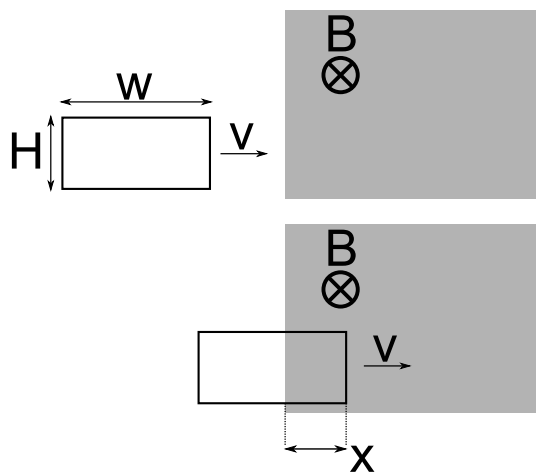
4. Find the average pressure that the light exerts on a perfectly absorbing particle of dust resting on the hemispherical dome. Give the analytical expression and numerical answer.

The average radiation pressure for a perfectly absorbing speck is given by the formula $\langle P_{\text{rad}} \rangle = \mathcal{I}/c = 9.07 \times 10^{-7} \text{ Pa}$.

Problem 3:

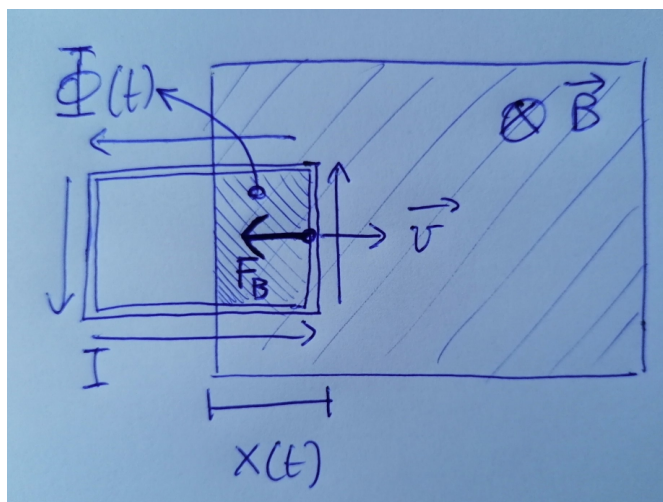
(5 pts)

A rectangular loop of wire of width w , height H and resistance R travels at a constant speed v into a uniform magnetic field B (grey region). The plane of the rectangular loop is perpendicular to the magnetic field, which is directed into the page. A schematic of the problem is shown below.



1. Find the magnitude of the induced current in the rectangular loop while it is entering the region with uniform magnetic field (as in the bottom panel of the figure above). Draw the direction of the induced current in the loop on a schematic (briefly qualitatively justify your answer). Clearly indicate the region where there is the uniform magnetic field on your drawing.

To solve this problem, we use the Faraday-Neumann-Lenz law. Due to Lenz law, the current is circulating counter-clockwise, in order to compensate the changing flux inside. This schematic is illustrated in the figure below



We first find the magnitude of the induced voltage by using the Faraday law

$$|V_{\text{ind}}| = \left| \frac{d\Phi}{dt} \right| = \left| \frac{d(BA)}{dt} \right| = \left| \frac{d(Hx)}{dt} \right| = BHv. \quad (11)$$

The current in the wire is then given by

$$I = \frac{V_{\text{ind}}}{R} = \frac{BHv}{R}. \quad (12)$$

- Find the magnitude of the total magnetic force exerted on the rectangular loop while it is entering the region with uniform magnetic field (as in the bottom panel of the figure above). Draw the direction of all the forces exerted on the loop on a schematic. Clearly indicate the region where there is the uniform magnetic field on your drawing.

The magnetic force is given by the formula (which derives from Faraday law applied to extended bodies) $\vec{F}_B = I\vec{L} \times \vec{B}$. The magnitude of the force can be computed as

$$F_B = IHB = \left(\frac{BHv}{R} \right) HB = \frac{B^2 H^2 v}{R} \quad (13)$$

The force is pointing inside the loop on the right vertical segment. Notice that the forces on the two horizontal segments sum up to zero.

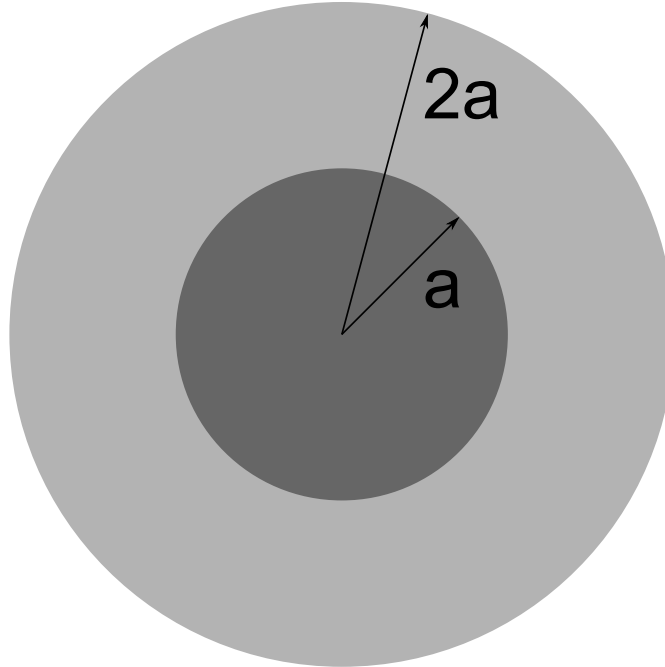
- Once the loop is entirely in the region of uniform magnetic field, what is the magnitude of the induced current ?

As the loop is entirely in the region of constant magnetic field, there is no flux change (since the enclosed area becomes constant as well). Because of Faraday law, the induced voltage is zero as well as the induced current.

Problem 4:

(11 pts)

A solid insulating plastic sphere of radius a (dark grey) carries a net positive charge $3Q$ uniformly distributed throughout its interior. The insulating sphere is coated with a metallic layer of inner radius a and outer radius $2a$ (light grey). The conducting metallic layer carries a net charge $-2Q$. The system is represent below



1. Find the volume charge density ρ associated with the charge uniformly distributed in the plastic sphere in terms of the variables introduced above.

The charge density is defined as $\rho = Q/V$, therefore

$$\rho = \frac{Q_{\text{tot}}}{V} = \frac{3Q}{\frac{4}{3}\pi a^3} = \frac{9Q}{4\pi a^3}. \quad (14)$$

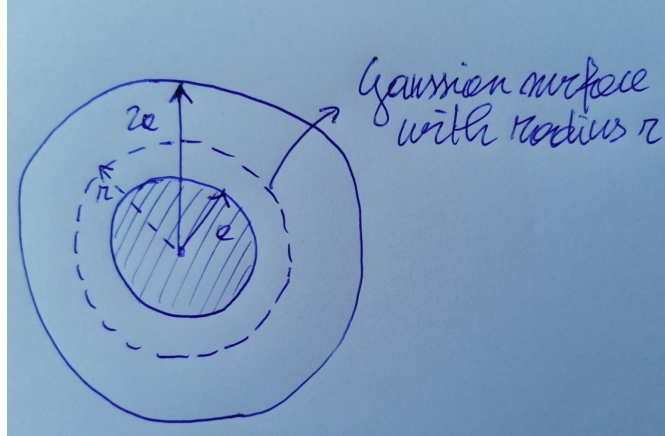
2. Apply Gauss law to derive the magnitude of the electric field in the different regions: (a) $r < a$ (b) $a < r < 2a$ and (c) $r > 2a$. For each region, draw the Gaussian surface you are using on the associated schematic, and indicate on the surface the direction of any vectors which appear in the mathematical expression of Gauss law. Express your answer in terms of a, Q, r and ϵ_0 .

(a) $r < a$

To solve this exercise, we apply the Gauss law,

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0}. \quad (15)$$

As a Gaussian surface, considering the symmetries of the problem, we can use a sphere of radius r . This schematic is illustrated in the figure below:



We hence have,

$$E4\pi r^2 = \frac{\rho V_{\text{enc}}}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{9Q}{4\pi a^3} \frac{4}{3}\pi r^3 = \frac{3Qr^3}{\epsilon_0 a^3}, \quad (16)$$

and we deduce the electric field for $r < a$

$$E = \frac{3Qr^3}{\epsilon_0 a^3} \frac{1}{4\pi r^2} = \frac{3Qr}{4\pi \epsilon_0 a^3}. \quad (17)$$

(b) $a < r < 2a$

the electric field is zero because the region is inside a conductor.

(c) $r > 2a$

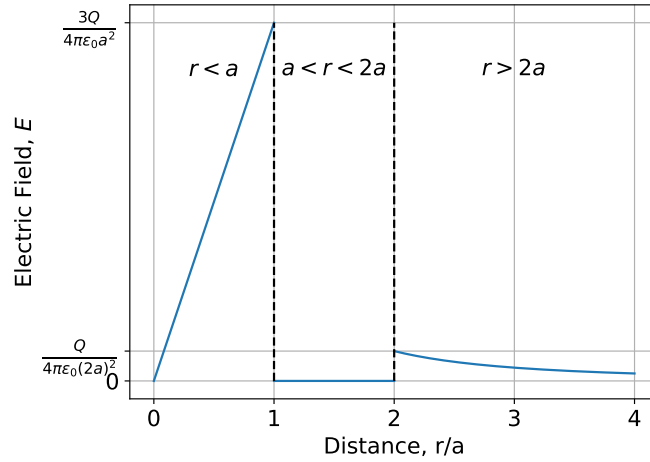
As in point (a) we apply Gauss' law (again considering a sphere as a Gaussian surface) and we get,

$$E4\pi r^2 = \frac{Q}{\epsilon_0}. \quad (18)$$

We therefore get $E = Q/4\pi\epsilon_0 r^2$, which is the electric field of a point-like charge.

3. Draw the electric field profile as a function of r in all different regions.

The field increases linearly in the region $r < a$, it is 0 in the region $a < r < 2a$ and it is decaying with $1/r^2$ in the region $r > 2a$.



4. Find the electric potential as a function of r in all regions. Use $V(\infty) = 0$ as the reference point.

(a) $r > 2a$

Taking $V(\infty) = 0$, we can start by deriving the potential in the region $r > 2a$. To do this, it is sufficient to integrate the electric field:

$$V(r) = V(\text{ref}) - \int_{\text{ref}}^r \vec{E} \cdot d\vec{r} = V(\infty) - \int_{\infty}^r \frac{Q}{4\pi\epsilon_0 r^2} dr = 0 + \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r} \right]_{\infty}^r = \frac{Q}{4\pi\epsilon_0 r}, \quad (19)$$

which is the electric potential of a point-like charge.

(b) $a < r < 2a$

To find the potential inside the conductor, we use the continuity argument $V(2a^-) = V(2a^+)$:

$$V(r) = V(\text{ref}) - \int_{\text{ref}}^r \vec{E} \cdot d\vec{r} = V(2a) - \int_{2a}^r 0 dr = \frac{Q}{4\pi\epsilon_0 2a}. \quad (20)$$

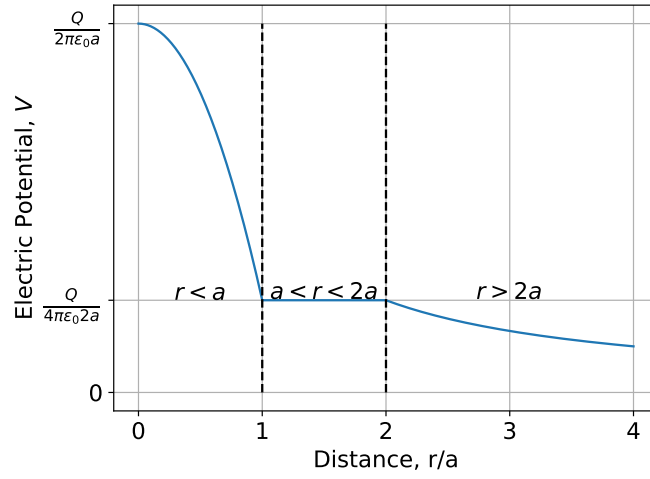
(c) $r < a$

Finally, to calculate the electric potential for $r < a$, we use both the continuity argument $V(a^-) = V(a^+)$, and we integrate the electric field

$$\begin{aligned} V(r) &= V(\text{ref}) - \int_{\text{ref}}^r \vec{E} \cdot d\vec{r} = V(a) - \int_a^r \frac{Qr}{4\pi\epsilon_0 a^3} dr = \frac{Q}{4\pi\epsilon_0 2a} - \frac{3Q}{4\pi\epsilon_0 a^3} \left[\frac{r^2}{2} \right]_a^r \\ &= \frac{Q}{8\pi\epsilon_0 a} \left[4 - \frac{3r^2}{a^2} \right]. \end{aligned} \quad (21)$$

5. Draw the electric potential as a function of r in all regions.

The electric potential decays with $-r^2$ in the region $r < a$, it is constant in the region $a < r < 2a$ and it decays with $1/r$ (point-like charge behavior) in the region $r > 2a$.



6. Find the net charge q_i residing on the inner surface of the metallic layer, and the net charge q_0 residing on the outer surface of the metallic layer.

Consider a Gaussian surface of radius $a < r < 2a$, we have,

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0} = 0, \quad (22)$$

because we are inside a conductor. It implies that,

$$Q_{\text{enc}} = q_i + Q_{\text{sphere}} = 0 \quad (23)$$

$$q_i = -3Q. \quad (24)$$

Hence, we have,

$$Q_{\text{conductor}} = q_i + q_0 \quad (25)$$

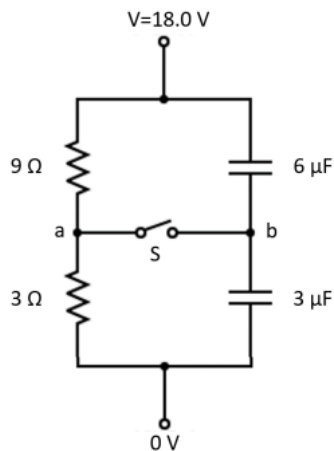
$$-2Q = -3Q + q_0 \quad (26)$$

$$q_0 = +Q. \quad (27)$$

Problem 5:

(8 pts), 3 questions

Consider the electrical circuit in the figure below.



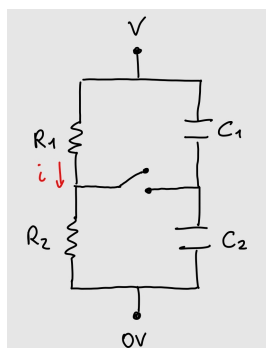
1. Calculate the potential difference between points a and b when the switch S is open. Write explicitly which of the two points is at a higher potential.

We start by calculating V_a . To do so, we evaluate the voltage across the resistors on the right branch of the circuit.

$$V - V_a = R_1 i, \quad (7)$$

$$V_a - 0 = R_2 i, \quad (8)$$

where we have used the naming convention as shown in the figure below.



The current i flowing through the two resistors is the same because they are in series and the switch is open. From the two equations above we can find the current:

$$i = \frac{V}{R_1 + R_2} = 1.5 \text{ A}. \quad (9)$$

And finally we find that $V_a = iR_2 = 4.5 \text{ V}$.

We now find V_b by calculating the voltage drops across the capacitors:

$$V - V_b = \frac{Q_1}{C_1}, \quad (10)$$

$$V_b - 0 = \frac{Q_2}{C_2}. \quad (11)$$

Since the capacitors are in series and the switch is open, they will have the same charge, so $Q_1 = Q_2 = Q$, and combining the two equations above we find:

$$V_b = \frac{C_1}{C_1 + C_2} V = 12 \text{ V}. \quad (12)$$

This results in a potential difference

$$V_b - V_a = 7.5 \text{ V},$$

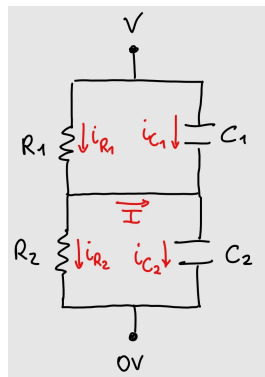
with $V_b > V_a$.

- At some point the switch S is closed. Calculate the value of the potential in point b a very long time after the switch is closed.

When S is closed, we have $V_a = V_b$ at all times. At $t = 0$, the charge in the capacitors is the same as when S was open, so V_b does not change, and $V_a = V_b = 12 \text{ V}$. Then, the charge on each capacitor begins to change. After a long time, at $t = \infty$, the circuit will reach a steady state where the charge on the capacitor will not change anymore. In that situation, the current flowing through the upper resistor is again the same one flowing through the bottom resistor, and hence it will still hold that $V_a = 4.5 \text{ V}$, as when S was open (see calculation in the question above). Furthermore, since $V_b = V_a$ at all times, we will have $V_b = 4.5 \text{ V}$.

- Write the Kirchhoff equations for the circuit at a generic time t after the switch S has been closed. Determine the characteristic time constant τ for the charge rebalancing between the two capacitors.

To find the equation that describes the dynamics of this system, we first use Kirchhoff's law. We define the currents at each node as shown in the figure below.



We apply Kirchhoff's current law:

$$i_{R_1} = i_{R_2} + I \quad (13)$$

$$i_{C_1} + I = i_{C_2} \quad (14)$$

and thus

$$i_{R_1} = i_{R_2} + i_{C_2} - i_{C_1}. \quad (15)$$

Next, using $V_a = V_b \equiv U(t)$, we can rewrite the current equation in terms of the voltages:

$$\frac{V - U(t)}{R_1} = \frac{U(t)}{R_2} + C_2 \frac{dV_{C_2}}{dt} - C_1 \frac{dV_{C_1}}{dt}, \quad (16)$$

where V_{C_1} and V_{C_2} are the voltages across each capacitor: $V_{C_1} = V - U$ and $V_{C_2} = U$.

Substituting these into the equation gives:

$$\frac{V - U(t)}{R_1} = \frac{U(t)}{R_2} + C_2 \frac{dU(t)}{dt} + C_1 \frac{dU(t)}{dt}. \quad (17)$$

Simplifying, we obtain:

$$(C_1 + C_2) \frac{dU}{dt} = \frac{V}{R_1} - \left(\frac{1}{R_1} + \frac{1}{R_2} \right) U. \quad (18)$$

We hence find an RC-like differential equation. We can solve it in analogy to how done in the lectures for an RC circuit:

$$(C_1 + C_2) \frac{dU}{dt} = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \left(V \frac{R_2}{R_1 + R_2} - U \right) \quad (19)$$

Let us define $R_{\text{eff}} = \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}$, $C_{\text{eff}} = C_1 + C_2$ and $V_{\text{eff}} = V \frac{R_2}{R_1 + R_2}$. We can then rewrite the above equation as:

$$\frac{dU}{U - V_{\text{eff}}} = - \frac{dt}{R_{\text{eff}} C_{\text{eff}}} \quad (20)$$

By integrating we find the solution:

$$U(t) = V_{\text{eff}} (1 - e^{-t/\tau}).$$

with

$$\tau = R_{\text{eff}} C_{\text{eff}} = \frac{C_1 + C_2}{\frac{1}{R_1} + \frac{1}{R_2}} = 20.25 \mu\text{s}. \quad (21)$$

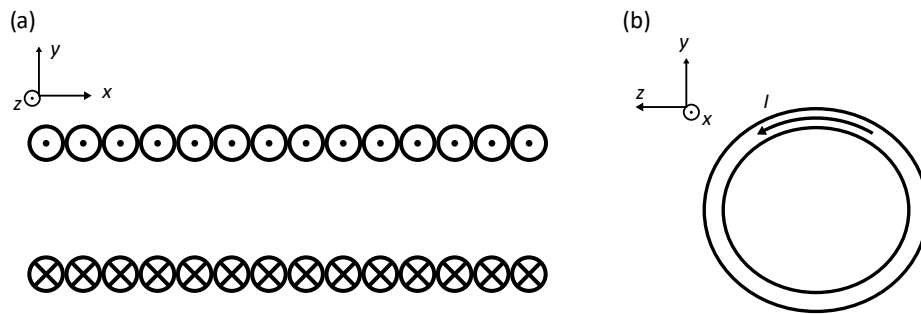
Note that for $t \rightarrow \infty$ the potential U reaches the value $V_{\text{eff}} = V \frac{R_2}{R_1 + R_2}$, which is exactly what found in question 2.

Problem 6:

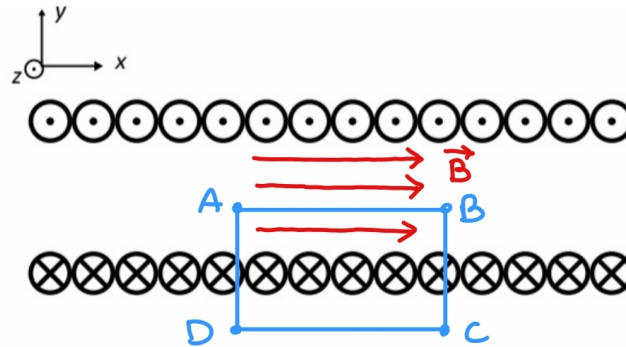
(7 pts)

An infinitely long ideal solenoid has n turns per unit length, and it is carrying a current I . A schematic of the cross section of the side view and the front view of this solenoid are respectively shown in panels (a) and (b) below.

1. Use Ampere's law to derive the magnitude of the field produced inside this solenoid. Draw a schematic of the orientation of the magnetic field inside the solenoid, the integration path used in Ampere's law and indicate any vectors which appear in the analytical expression of Ampere's law.



The field outside an infinite solenoid is zero, while inside the solenoid the field is parallel to the solenoid's axis (x direction). To compute the magnetic field inside the solenoid we choose a rectangular loop passing through one side of the solenoid, as shown in the figure below.



By applying Ampere's law on this loop we find:

$$\mu_0 \sum I = \oint \vec{B} \cdot d\vec{l} = \int_A^B \vec{B} \cdot d\vec{l} + \int_B^C \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l} + \int_D^A \vec{B} \cdot d\vec{l} \quad (28)$$

In the segment CD we have that $\vec{B} = 0$. In the segments BC and DA the field is perpendicular to $d\vec{l}$, so they do not contribute to the integral. Finally, in the segment AB the field is constant and parallel to $d\vec{l}$, so we find

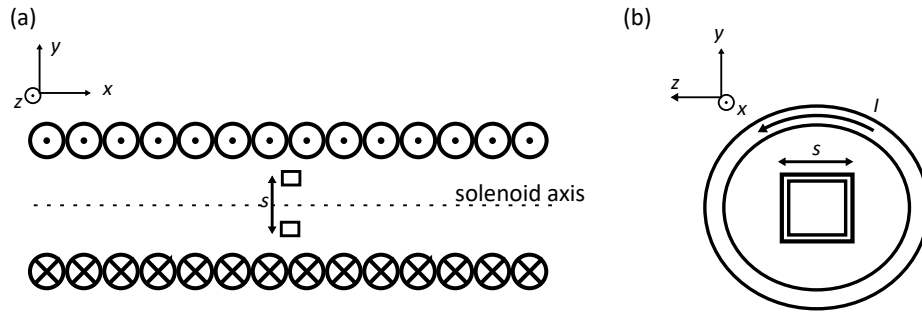
$$\mu_0 \sum I_{\text{enc}} = \oint \vec{B} \cdot d\vec{l} = B \int_A^B d\vec{l} = BL \quad (29)$$

Where L is the length of the segment AB . The current enclosed by the loop is $\sum I_{\text{enc}} = nLI$ where n is the number of turn per meter and I the current flowing in the solenoid. Finally, we find:

$$B = \mu_0 n I_{\text{sol}}. \quad (30)$$

2. The infinitely long ideal solenoid has $n = 400$ turns per meter. The current flowing through the solenoid is described by the function $I = 3t^2$, where t represents time in seconds and I is the current in amperes. A square coil with sides of length $s = 1.3$ cm is fixed inside the solenoid such that its axis coincides with that of the solenoid, as depicted in the figure below. At $t = 0.75$ seconds, the induced electromotive force (EMF) is $22.2 \mu\text{V}$. A schematic of the cross section of the side view and the front view of the solenoid with the coil inside are respectively shown in panels (a) and (b) below.

Find the number of turns of the square coil. Note that the thickness of the wires can be neglected throughout the problem. Give the analytical expression and numerical answer.



The magnetic field generate by the solenoid is

$$B = \mu_0 n I(t) = \mu_0 n 3t^2 \quad (31)$$

The flux through the square coil is

$$\Phi = N B A_{\text{coil}} = N \mu_0 n 3t^2 s^2, \quad (32)$$

where N is the number of turns of the square coil.

The induced EMF can be computed from Faraday's law as:

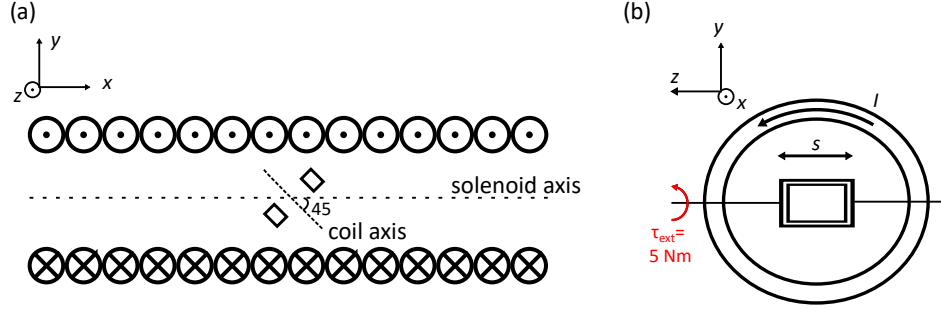
$$|\mathcal{E}_{\text{ind}}| = \left| \frac{d\Phi}{dt} \right| = N \mu_0 n 6ts^2 \quad (33)$$

We can hence calculate the number of turns N :

$$N = \frac{\mathcal{E}_{\text{ind}}}{\mu_0 n 6ts^2} = 58. \quad (34)$$

3. The current flowing through the solenoid is now constant in time and has a magnitude of $I_{\text{sol}} = 10 \text{ mA}$. The square coil within the solenoid is now attached such that it can rotate around an axis aligned along the z direction and passing through its center. An external torque of $\vec{\tau}_{\text{ext}} = 5 \text{ N m} \hat{z}$ is applied on the coil.

If the square coil axis is at a 45° angle relative to that of the solenoid, find the required current that must flow through the square coil and its direction to prevent any rotational movement of the square coil. Recall that the solenoid has 400 turns per meter, the square coil has sides of length $s = 1.3 \text{ cm}$. For the number of turns in the square coil, use your answer to the question above. Give the analytical expression and numerical answer.



There are two torques applied on the square coil. The external torque and a magnetic torque created by the field. The magnetic torque is:

$$\vec{\tau}_{\text{mag}} = \vec{\mu} \times \vec{B}. \quad (35)$$

where the magnetic moment of the coil is given by

$$\vec{\mu} = NI_{\text{coil}} A_{\text{coil}} \hat{u}_n = NI_{\text{coil}} s^2 \hat{u}_n, \quad (36)$$

where $\hat{u}_n$ is the unit vector perpendicular to the plane of the coil. Note that this vector's direction depends on the angle of the coil. If we call $\theta = \pi/4$ the angle between the magnetic field inside the solenoid (which is along the x axis) and $\hat{u}_n$, then the intensity of the magnetic torque is:

$$\vec{\tau}_{\text{mag}} = NI_{\text{coil}} s^2 B \sin(\theta) \hat{z} = NI_{\text{coil}} s^2 \mu_0 n I_{\text{sol}} \sin(\theta) \hat{z} \quad (37)$$

In order for the coil to be still, the total torque must be zero. This means that the external torque and the magnetic torque must be equal in intensity and opposite in direction.

$$|\vec{\tau}_{\text{ext}}| = |\vec{\tau}_{\text{mag}}| \quad (38)$$

$$I_{\text{coil}} = \frac{\tau_{\text{ext}}}{NI_{\text{sol}} s^2 \mu_0 n \sin(\theta)} = 1.48 \times 10^8 \text{ A}. \quad (39)$$

We still need to find the direction of the current. Since the external torque is directed along the positive z direction, then the magnetic force must be directed along the negative z direction. For this to the case, the current in the coil must be flowing clockwise, as depicted in the figure below.

